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Random packing of lines in a lattice cube

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A study is made of the random sequential packing of complete lines in a cube of integer lattice points, with side N. For $N \leq 15$ exact packing fractions are computed. It is found that if line occupation attempts arrive as a spatial Poisson process the packing has two distinct phases; initially where large numbers of potential adsorption sites are blocked, and subsequently where no further blocking occurs so that filling is exponential in time. It is shown that the ratio of the durations of the blocking to the nonblocking phases falls to zero as $N \to \infty$. In this limit, the packing fraction at time t is $\theta(t) = \frac{3}{4}(1-e^{-t})$. The rapid switch between phases in large systems creates a dramatic fall in the packing rate at the start of the process. This becomes a discontinuity as $N \to \infty$ and is a consequence of the high aspect ratio of the packing objects. It provides a physical explanation for the diverging coefficients in expansions of $\theta(t)$ about $t=0$ for objects with diverging aspect ratio. After considering the three-dimensional case, the analysis is extended to d-dimensional cubes, for which it is conjectured that $\theta = d/2^{d-1}$ in the limit $N \to \infty$.

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I. INTRODUCTION

A great variety of structures are created by objects packed together. The nature of these structures has been of interest to people for centuries at least. In the last century, packing problems have appeared abundantly in many different pure [1,2] and applied fields [3–6], and the variety of packing processes, object types, and spaces is endless. However, due to the enormous combinatorial problems that such systems pose, few can be solved exactly.

The work in this paper is motivated by a desire to investigate the packing of rodlike structures and particularly those with large aspect ratio [7–11]. A three-dimensional packing problem is considered whose evolution may be described by a recursion relation solvable exactly by computer for sufficiently small systems. For large systems an approximate solution is obtained which becomes exact in the large size limit. The packing occurs in a cube of integer lattice points, with side N. We consider two closely related formulations. The first: a line of N unoccupied nearest-neighbor lattice points is chosen within the cube, and occupied. The process is repeated until no more such lines exist. The fraction of occupied points at this stage is the packing fraction. The second formulation is as a spatial Poisson process: each line of lattice points within the cube is subject to occupation attempts at a fixed rate. An occupation attempt fails if any of the points of the line are already occupied. An equivalent picture of the process is to imagine attempts to thread thin rods into the cube, normal to the surface, along lines of lattice points. If any point along the line is already occupied then the attempt fails and a new insertion point is chosen. As time progresses, the lattice fills up, and the success rate of occupation attempts declines, falling to zero when the lattice is saturated (jam packed).

A problem equivalent to our first formulation was considered by Isokawa [12] whose simulation results suggested a packing density tending to 3/4 for large systems. Using a cumulant truncation method, we confirm this. The method allows us to relate the kinetics of the packing process to the cube size, and provides the time dependence in the large size limit. As mentioned above, we present an exact solution method for small cubes. The method involves numerical solution of a recursive evolution equation which is equivalent to the simulation algorithm presented in [12]. We also extend our study to higher dimensional cubes. The filling of complete lattice lines is of practical interest because it provides a model for the packing of particles with high aspect ratio. The process is distinct from the packing of particles with high aspect ratio into a space significantly larger than their length. However long-range blocking effects are common to both processes and the behavior of our system is driven by this effect. The packing of lattice lines is analogous to the random packing of fibers into a reinforced composite matrix [13]. However, our main interest in this model is mathematical in contrast to more experimentally motivated and practical models of rod packing [7]. Other examples of fiber packings are cellulose fibers in paper and colloidal rods in sediments.

The most successful early solutions of packing problems focused on the one-dimensional case. Renyi's solution to his classic problem of random car parking [14] made use of the fact that when a car parks on a road, two shorter roads are created which are in all other respects equivalent to the original. This leads to a recursive solution recently generalized to a distribution of car lengths [15]. In higher dimensions, the problem is more difficult because the arrival of an object merely creates a space with a more complicated structure. The standard and much studied approach to solving such problems is to construct a rate equation for the decay of the probability that a single test point remains empty at time t. For the point to be occupied, it must form part of an object-shaped gap, and therefore its rate equation must depend on the probability that such a gap exists. The nonoccupation probabilities of these gaps in turn obey their own rate equations which depend on the states of larger regions still, and an infinite hierarchy of rate equations emerges. These are

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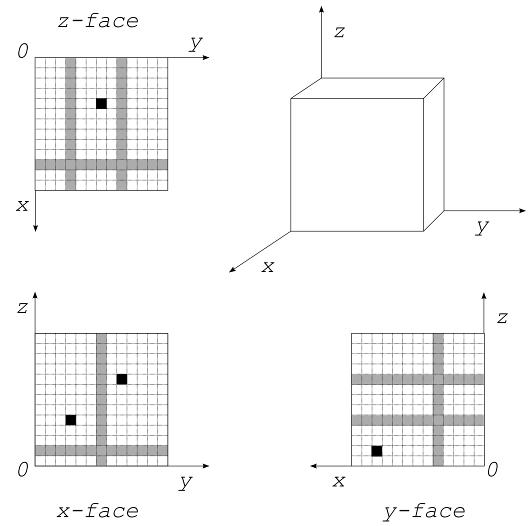

FIG. 1. A lattice cube with four complete lines occupied. The three grids represent three faces of the cube which are normal to the positive coordinate directions. The dark squares represent the ends of occupied lines and the gray lines represent sites which are blocked to occupation by the existing occupied lines. The state of this cube may be written as follows: $\{f_{xy}=1, f_{xz}=1, f_{yx}=2, f_{yz}=1, f_{zy}=1, f_{zx}=2\}$.

rarely exactly solvable and lead to divergent series expansions usually in powers of the time for which the process has been running [16]. Physically motivated truncation or resummation methods are then required to yield results [3,17]. Particles with very large aspect ratios pose particular problems for this technique, because the regions described by the rate equation hierarchy grow rapidly, and so therefore do the magnitudes of the series expansion coefficients. We suggest that this mathematical behavior reflects the physical behavior exhibited by the system studied here: during the initial phase of packing, many potential landing sites are blocked by each filling event. Eventually a set of gaps in the packed structure remains, the filling of each of which serves only to block itself and no others. This noninteracting packing process leads to a packing rate which is exponential in time. For large systems, the interactive blocking phase is short and the packing rate during this phase falls steeply creating a kink in $\dot{\theta}(t)$. Series expansions of this function about $t=0$ will therefore contain divergent coefficients and have a narrow radius of convergence. This, together with the kink, causes the standard solution techniques to fail.

II. RECURSIVE COMPUTATION OF THE PACKING FRACTION

In this section we examine the first formulation of the process where a line is filled with certainty at each step until the lattice is packed. We begin by observing that when a line of points within an empty cube becomes occupied, a line of points on each of the four adjacent faces becomes blocked to further threading attempts. We will refer to these blocked lines as *face-lines* (see Fig. 1). Further occupation of internal lines will create more face-lines, but as the cube fills up, these are more likely to overlap existing face-lines and therefore produce no further blocking effect. Eventually, the arrangement of face-lines will be such that further filling of internal lines, if possible, serves only to prevent the adsorption of another internal line in the same place. For larger cubes, such configurations tend to be reached earlier in the packing process; after this all remaining surface sites will be threaded with certainty.

The important quantities in the problem are the number of face-lines on each surface of the cube. We need only consider three of the six cube faces, because opposite faces will have identical configurations of face-lines. We will label the Cartesian directions $\{x,y,z\}$. Let f_{ij} be the number of face-lines oriented in the j direction on the face whose outward normal is in the positive i direction; the i face. There are six f_{ij} corresponding to the possible permutations of two directions $\{i,j\}$ drawn from the set $\{x,y,z\}$. Figure 1 is an example of an early stage cube configuration. We note the following inequality which must hold at all stages of the process:

$$f_{ij}+f_{ji} \le N. \tag{1}$$

It holds because face-lines can never form a loop around the cube. Such a loop would imply the intersection of two internal lines of occupied points. It follows from Eq. (1) that in a cube with occupied lines in all three orientations the maximum number of face-lines is $3N$. No more face-lines can be created by further occupations. In cubes with occupied lines oriented only in the ith direction, $f_{ij}=f_{ik}=0$ so the maximum number of face-lines is $2N$.

We will now show how the packing fraction depends on the number of face-lines. Consider the x face. The number of points on the face which are not part of face-lines is $N^2 - N(f_{xy}+f_{xz})+f_{xy}f_{xz}$. Once the cube has reached the stage where no more face-lines can be created by threading more internal lines, we know that surface points not blocked by face-lines have either been threaded already or will be. The packing fraction is therefore

$$\theta(\{f_{ij}\}) = \frac{1}{N^2}\sum_{\langle i,j,k\rangle} N^2 - N(f_{ij}+f_{ik}) + f_{ij}f_{ik}, \tag{2}$$

where $\langle i,j,k\rangle$ stands for all even permutations of $\{x,y,z\}$. We are interested in computing $\mathbb{E}[\theta(\{f_{ij}\})]$, where the expectation is over the final configuration of face-lines (when no more can be created). To compute this we must find an equation that describes the evolution of the numbers $\{f_{ij}\}$ through the sequence of line occupations. However, before we do so, we note that our inequality in Eq. (1) immediately suggests an approximate solution, which we now examine.

A. Approximate solution

Let us suppose that inequality (1) is strict for some pair, (i,j), of directions: $f_{ij}+f_{ji}<N$, and that there is at least one occupied internal line in each direction. This second condition ensures that the cube contains no complete planes of occupied sites, which would completely block any further occupations of lines perpendicular to them. If the conditions are satisfied there must be at least one layer of sites, forming

an (i,j) plane, which contains no occupied lines in either the i or the j direction, and within which it must be possible to occupy at least one complete line. This occupation will create one more face-line. Provided the inequality still holds strictly, the same argument may be repeated until it becomes an equality. The argument applies equally to all pairs of directions, so under the assumption that there is at least one occupied line in each direction, the number of face-lines must eventually reach its maximum of $3N$. If the assumption is then made that rod orientations are distributed equally amongst the three Cartesian directions, the final number of face-lines in each direction on each face will be $N/2$. We will see later that as $N\to\infty$ this becomes a progressively closer approximation to the truth. Substituting this value ($f_{ij}=N/2 \ \ \forall \ \ i,j$) into our expression (2), we find that $\theta=\frac{3}{4}$. This turns out to be the correct result in the limit $N\to\infty$, but does not hold for finite cubes where early fluctuations away from an even distribution of rod orientations have a greater effect.

B. Exact solution

We now turn to the exact analysis of finite cubes. We know that each additional internal line of occupied sites will produce a new face-line on one or both adjacent faces, or on neither. The number of ways that each of these outcomes can occur is referred to as its valency, ν. The creation of a face-line on both adjacent faces due to the occupation of an internal line in the ith direction has valency

$$\nu[\{f_{ki},f_{ji}\}\to\{f_{ki}+1,f_{ji}+1\}]=(N-f_{ik}-f_{ki})(N-f_{ij}-f_{ji}), \tag{3}$$

where j and k represent the other two directions and $\{i,j,k\}$ is an even permutation of $\{x,y,z\}$. There are two more equivalent valencies corresponding to the occupation of an internal line in each of the remaining directions. Expressions for these are obtained via even permutation of the indices, corresponding to rotations which permute all faces, not just two. The quantity $N-f_{ik}-f_{ki}$ in Eq. (3) is the number of lines of surface points in the k direction which are not covered by k oriented face-lines and adsorption through which would produce a new blocked face-line in the i direction on the k face. The quantity $N-f_{ij}-f_{ji}$ has an identical meaning but with $k\leftrightarrow j$. The product of the two is therefore the number of insertion points which would give a new face-line on both adjacent faces.

The addition of a single face-line on one adjacent face (in the ith direction on the k face here) has valency

$$\nu[f_{ki}\to f_{ki}+1]=(N-f_{ik}-f_{ki})f_{ji}. \tag{4}$$

There are six such valencies corresponding to the six $\{f_{ij}\}$. These may be obtained via permutations, odd and even, of the indices. Equation (4) takes this form because to only add a face-line to the k face, the face-line on the j face must overlap a pre-existing line, of which there are f_{ji}.

If n internal lines have been occupied, the number of possible occupation events which produce no change in the face-line configuration is

TABLE I. Packing fractions for cubes up to size 15 expressed correct to 20 decimal places.

Cube side (N)	Packing fraction (θ)
1	1
2	0.95238095238095238095
3	0.89153158569825236492
4	0.85335057657291153834
5	0.82971764390022581363
6	0.81436937584536562651
7	0.80382687649629625433
8	0.79621032124966601065
9	0.79047211353996789033
10	0.78600022786064020440
11	0.78241895186861584334
12	0.77948673942795019983
13	0.77704182683303941979
14	0.77497199350954047158
15	0.77319700252486999064

$$\nu[\varnothing]=\sum_{\langle i,j,k\rangle} f_{ji}f_{ki}-n. \tag{5}$$

Summing all valencies we obtain the total

$$V[(f_{ij}),n]=\left\{\sum_{\langle i,j,k\rangle}[N^2-N(f_{ij}+f_{ik})+f_{ij}f_{ik}]\right\}-n, \tag{6}$$

which we will use as a normalizing factor. Letting f stand for the configuration of face-lines then defining

$$\omega(f'\to f)=\frac{\nu[f'\to f]}{V[f',n]} \tag{7}$$

we have that the probability of any given configuration after adding n rods obeys

$$p(n,f)=\sum_{f'}\omega(f'\to f)p(n-1,f'), \tag{8}$$

subject to the boundary condition $p(0,f_0)=1$ where f_0 corresponds to the empty cube (all $f_{ij}=0$). This recursion equation may be solved on a computer either exactly or to arbitrary precision using hash tables to store the configuration probabilities $p(n,f)$ at each step. A configuration f, occurring at step n^* is terminal if n^* is the smallest integer such that $V(f,n^*)=0$. The mean packing fraction is then

$$\theta=\sum_f p(n^*,f)\theta(f). \tag{9}$$

Table I lists the packing fractions computed for cubes up to size 15. Computing θ for $N=15$ takes about 1 h on a modern PC. The CPU time approximately doubles for each increment of N. The packing fractions are plotted in Fig. 2. For smaller cubes the fractions are greater because the rods have a tendency to align and therefore fill space more efficiency. As the cube size increases the distribution of alignments becomes more uniform and the packing fraction tends toward

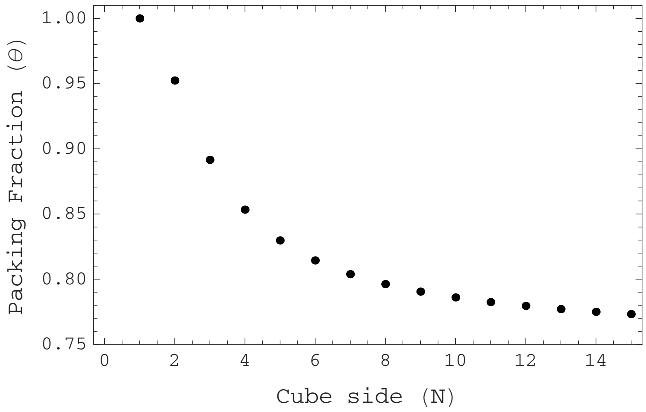

FIG. 2. Packing fractions for cubes up to size $N=15$.

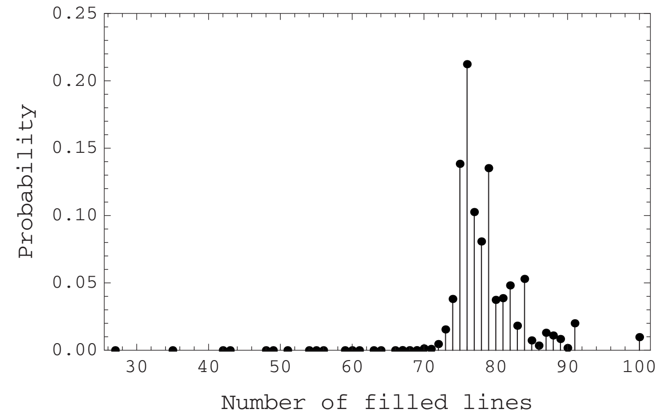

FIG. 4. Exact probability mass function for the number of occupied lines in an $N=10$ cube.

its infinite cube value of 3/4. Figures 3–5 show probability mass functions for the numbers of occupied lines in the jammed state for $N=5$, 10, and 15. Two points are of interest here: first, there exist jammed states with very low packing fraction, the smallest having $\theta=3(N-1)/N^2$. Second, the variance of the packing fraction narrows as N increases.

III. CUMULANT TRUNCATION APPROACH FOR LARGE N

We now turn to the second formulation of the problem in terms of a spatial Poisson process. A well developed set of methods exists [3,17–19] for constructing series expansions for the packing fraction in powers of the time, or other well chosen function of the time, about $t=0$. The application of these methods to packing processes involving particles with a large aspect ratio leads to series expansions whose coefficients diverge with this ratio. Due to the simple geometry of our system, an alternative approximate approach is open to us which becomes exact in the large N limit. The behavior of the system in this limit will provide a physical explanation for the diverging coefficients.

We will consider the expectation of the number f_{ij} at time t,

$$\langle f_{ij}\rangle(t) \equiv \mathbb{E}[f_{ij}(t)]. \tag{10}$$

We assume that every line of lattice points is subject to occupation attempts at unit rate. The rate of increase of f_{ij} is therefore proportional to the number of ways that an unoccupied line can be filled to produce a new face-line in the jth direction on the ith face. This number is

$$\nu_{ij} = N^2 - N(f_{ij}+f_{ji}+f_{jk}) + f_{jk}(f_{ij}+f_{ji}). \tag{11}$$

It is derived by adding together the two valencies which correspond to an increment in f_{ij}. We, therefore, have that

$$\mathbb{P}[f_{ij}(t+\delta t) - f_{ij}(t) = n|\mathcal{F}_t] = \begin{cases} \nu_{ij}\delta t & \text{if } n=1 \\ 1-\nu_{ij}\delta t & \text{if } n=0 \\ 0 & \text{otherwise,} \end{cases} \tag{12}$$

where $\mathcal{F}_t$ represents the state of the cube at time t. From this we see that

$$\mathbb{E}\left[\frac{d}{dt}f_{ij}|\mathcal{F}_t\right] = \nu_{ij}. \tag{13}$$

Taking expectations conditional on the initial state $\mathcal{F}_0$ we get

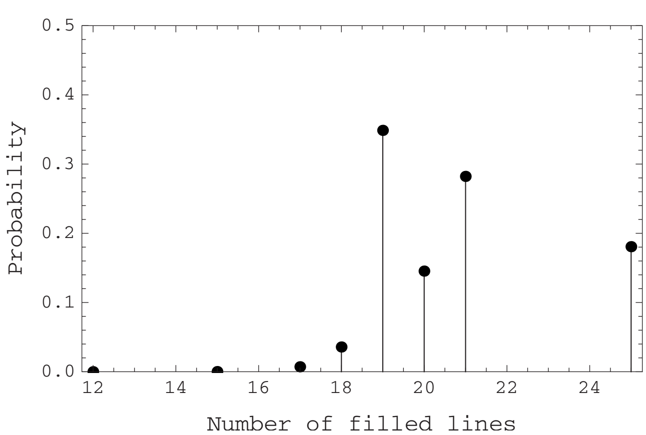

FIG. 3. Exact probability mass function for the number of occupied lines in an $N=5$ cube.

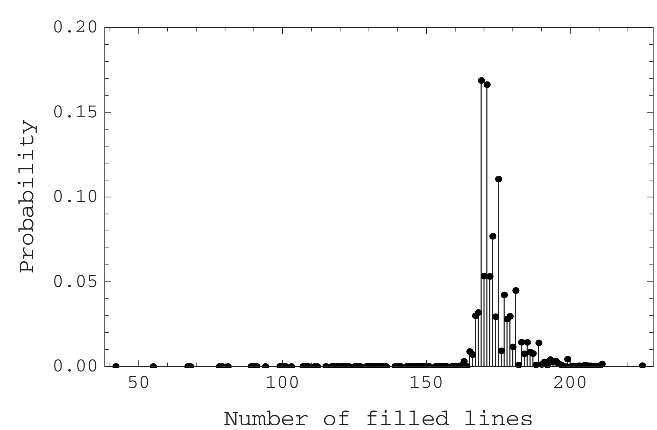

FIG. 5. Exact probability mass function for the number of occupied lines in an $N=15$ cube.

$$\frac{d}{dt}\langle f_{ij}\rangle = N^2 - N[\langle f_{ij}\rangle + \langle f_{ji}\rangle + \langle f_{jk}\rangle] + \langle f_{jk}f_{ij}\rangle + \langle f_{jk}f_{ji}\rangle \tag{14}$$

subject to the boundary condition $\langle f_{ij}\rangle(0)=0$. There are six equations of this kind obtained from Eq. (14) by setting $\{i,j,k\}$ equal to the permutations of $\{x,y,z\}$. These equations are exact in the sense that they contain no approximations. However, they cannot be solved on their own because of the moments $\langle f_{jk}f_{ij}\rangle$ and $\langle f_{jk}f_{ji}\rangle$, which themselves satisfy first-order ordinary differential equations (ODEs) which depend on third moments and so on. To make progress we will need to truncate this hierarchy.

We introduce the simplest possible truncation scheme which we argue becomes valid in the limit $N\to\infty$. The approximation is

$$\langle f_{jk}f_{ij}\rangle = \langle f_{jk}\rangle\langle f_{ij}\rangle, \tag{15}$$

$$\langle f_{jk}f_{ji}\rangle = \langle f_{jk}\rangle\langle f_{ji}\rangle, \tag{16}$$

which is equivalent to the statement that the six f_{ij} are uncorrelated, or that the cumulant of any pair is insignificant compared to the magnitude of the numbers f_{ij}. We now justify this for large N: in smaller cubes lines occupied early on have greater blocking effect because the fraction of all lattice lines newly blocked by each early occupation is proportional to $1/N$. In the long run, occupation attempts are equally distributed amongst the three possible orientations of internal lines. Fluctuations away from this even distribution after n occupation attempts will be proportional to $1/n$, and whilst $n\ll N$ almost all of these attempts will be successful. Therefore, in small cubes early fluctuations in the distribution of occupied lines have a more pronounced effect on the directional distribution of further occupations. As the size of cube tends to infinity, the number of occupations required to block a significant fraction of face-lines also tends to infinity. This means that in large cubes when n is small, the f_{ij} will be uncorrelated because an excess of one type of face-line, caused by random fluctuations, has insignificant blocking effect and cannot therefore influence the directional distribution of further occupations. Once enough rods have been added to create a blocking effect sufficient to influence the fraction of successful occupations in a particular direction, fluctuations away from an even distribution of occupied line directions have been destroyed due to the large number of filling events. It must therefore be the case that in the limit $N\to\infty$, for every realization of the process, as $n\to\infty$ the ratios $f:f'\to 1:1$ where f and f' are any two face-line numbers. If all the face-line numbers are in equal ratio for every realization of the process then they are not correlated.

If the distribution of face-lines for each realization of the process evens out as the cube size increases then we should expect the probability distribution of packing fractions to narrow. This effect is illustrated in Figs. 3–5 which show the probability mass functions for the numbers of occupied lines for cubes of size 5, 10, and 15. The variance in the packing fraction for smaller cubes is high because the early stages of the filling process have a stronger influence on the orientations of later successful filling attempts. As the cube size increases the distribution becomes more sharply peaked. As $N\to\infty$ we would expect the distribution to tend toward a delta function.

It is clear from the symmetry of the cube and the filling process that all the $\langle f_{ij}\rangle$ must be equal, and this symmetry is present in the set of ODEs. Defining $\bar{f}=\langle f_{ij}\rangle$, using our approximation we have

$$\frac{d}{dt}\bar{f} = N^2 - 3N\bar{f} + 2\bar{f}^2. \tag{17}$$

Solving subject to $\bar{f}(0)=0$, we find

$$\bar{f} = \frac{N(1-e^{-Nt})}{2-e^{-Nt}}, \tag{18}$$

from which we see that as N becomes larger $\bar{f}$ tends increasingly rapidly to the constant $N/2$. At this point no more face-lines can be created and the blocking phase is over. Because the duration of this phase tends to zero with increasing system size, so does the volume fraction occupied during it. In the large N limit, $\bar{f}\to N/2$ instantaneously and the fraction of sites occupied during the blocking phase is zero. The number of completely unoccupied lines left inside the cube at this point is $3N^2/4$ and since the filling of these lines produces no further blocking effect they will all become filled: $\theta=3/4$. The rate of packing for any size of cube and at any stage of the process will be proportional to the fraction of nonblocked surface sites remaining (sites not covered by face-lines). Let this fraction be $X(t)$. If each new line blocks only the surface site through which it was threaded, then $\dot{X}=-X$, so that $X(t)=X(t^*)e^{-(t-t^*)}$, where t^* is the time at which the blocking phase ends. We will refer to $t>t^*$ as the exponential phase. As $N\to\infty$ we find that $t^*\to 0$ and $X(t^*)\to 1/4$, so that $X(t)=(1/4)e^{-t}$. In a cube of size N, if each surface point is subject to threading attempts at unit rate then the probability of filling a new line in time δt is $3N^2X(t)\delta t$, so $\dot{\theta}(t)=3X(t)$. The packing fraction at time $t>0$ in the $N\to\infty$ limit is therefore

$$\theta(t) = \frac{3}{4}(1-e^{-t}). \tag{19}$$

To reiterate, this equation holds in the limit $N\to\infty$ because the duration of the blocking phase tends to zero so the system immediately enters the exponential phase. A finite system, however, whatever its size, will have a finite blocking phase. For all systems, at $t=0$, the lattice is empty $[X(0)=1]$, so the packing rate $\dot{\theta}(0)=3$. For large systems, at the start of the exponential phase the packing rate is close to 3/4, so as $N\to\infty$ the packing rate $\dot{\theta}(t)$ falls increasingly rapidly from 3 to $\approx 3/4$ at the start of the process. The rate therefore has a kink which turns into a discontinuity as $N\to\infty$. The sharpening of this kink is illustrated in Fig. 6, which shows expected packing rates obtained from simulation. Series expansions of $\dot{\theta}(t)$ about $t=0$ will contain coefficients which diverge as N increases and the discontinuity appears. The radii of convergence of such expansions will tend to zero.

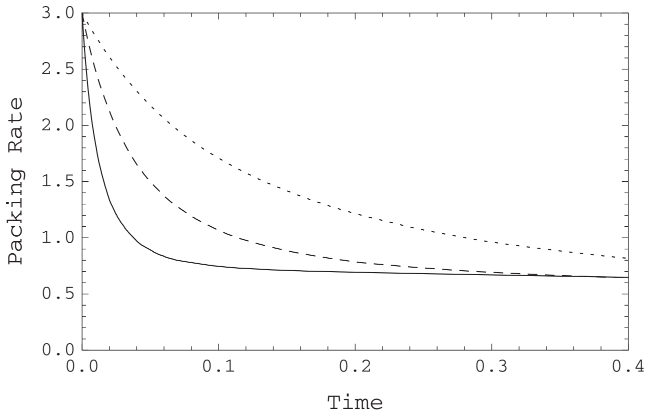

FIG. 6. Simulation results for the mean early packing rates in an $N=10$ (dotted line), $N=30$ (dashed line), and $N=100$ (full line) cube. Each surface site is subject to threading attempts at unit rate. For larger cubes the kink in the packing rate marking the transition from the blocking to exponential phase becomes more distinct.

The kink prevents the use of standard techniques such as the Padé approximation [18–20] normally used to extract long time behavior from expansions derived for conventional packing problems.

IV. CUBES OF HIGHER DIMENSION

The methods used to treat three-dimensional cubes may be extended to cubes of arbitrary dimension, provided we assume an even distribution of packed line orientations. This assumption was justified in the previous section for large cubes and a similar argument may be applied in higher dimensions. We will consider the four-dimensional case first and then generalize. A four-dimensional cube of side N consists of the set of integer lattice points $\{\vec{r} \equiv (w,x,y,z,)|w,x,y,z \in \{0,1,\ldots,N-1\}\}$. We will refer to the cube faces as the subsets of points created by setting one of the coordinates equal to $N-1$. For example, the w face consists of the set of points $\{(N-1,x,y,z)|x,y,z \in \{0,1,\ldots,N-1\}\}$. As mentioned previously, we need to consider only one of each pair of opposite faces.

A line of lattice points in the w direction within the cube may be described by the equation

$$\vec{r} = \begin{pmatrix} 0 \\ x^* \\ y^* \\ z^* \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{20}$$

where $\lambda \in \{0,1,\ldots,N-1\}$. As with the three-dimensional case, occupying such a line can be thought of as the insertion of a rod through the w face at the point $(N-1,x^*,y^*,z^*)$. We now ask: which insertion points within the other faces are blocked by the occupation of this line? The condition for successful occupation of a lattice line is that it does not intersect any existing occupied lines. For example, the x-orientated line

$$\vec{r} = \begin{pmatrix} w^* \\ 0 \\ y^* \\ z^* \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \tag{21}$$

is disallowed for any choice of w^*. This means that the set of points within the x face (a cube) with $y=y^*$ and $z=z^*$, which is a line in the w direction, is a face-line (insertion attempts through any point on it will fail). Similarly, the line $x=x^*$, $z=z^*$ within the y face and the line $x=x^*$, $y=y^*$ within the z face are also face-lines. In the three-dimensional case, we argued that face-lines could not form a loop around the cube. We now need to find the equivalent condition for four dimensional cubes. Let us consider a face-line $z=z^*$, $w=w^*$ within the y face. Inserting a rod through any of the points on this line would create the face-line $z=z^*$, $w=w^*$ in the x face; therefore, since all such insertion points are disallowed no such line can ever be created. It follows that no two faces can have face-lines with the same two constant coordinates. Every pair of constant coordinates defines two lines, one within each of the two faces corresponding to the remaining two coordinates. Only one of these two lines can be a face-line (blocked to insertion attempts). This is the four-dimensional equivalent of the no loops condition in three dimensions. The d-dimensional equivalent is as follows: in a d-dimensional cube, each face is of dimension $d-1$. Every $(d-2)$-tuple of constant coordinates defines two lines, one in each of the two faces corresponding to the two remaining coordinates. Only one of these can be a face-line.

We now use this condition to determine the packing density. Within each face there are $d-1$ possible orientations for face-lines. If N is sufficiently large so that the orientations of packed internal lines are distributed evenly amongst the directions, then since the face-line defined by a given $(d-2)$-tuple of constant coordinates cannot appear twice in two different faces, then only half of all the possible lines in any given direction within a face will be face-lines. Let us now consider a test point within a face. Let p be the probability that the point is not intersected by a face-line. If there is no correlation between the number of face-lines in each direction, which we have argued must be the case as $N \to \infty$, then once no more face-lines can be created $p=(1/2)^{d-1}$. If the point is not on a face-line then a rod must eventually be inserted through it, occupying N sites within the cube. Within each of the d faces there are N^{d-1} points so the total volume of occupied sites within the cube will be $N^d(1/2)^{d-1}d$. This corresponds to a packing fraction

$$\theta_d = \frac{d}{2^{d-1}}. \tag{22}$$

For $d=3$ we recover our previous result: $\theta_3=\frac{3}{4}$. It may be noted that the definition of a four-dimensional cube may be generalized to $d<3$ and that $\theta_1=\theta_2=1$, which is the correct result in these two cases.

V. CONCLUSIONS

We have investigated the random packing of lines in a lattice cube in three and higher dimensions. Exact packing fractions were computed for small cubes in three dimensions and asymptotic packing fractions in arbitrary dimension. The time dependence of the packing fraction in large cubes was described. It was discovered that a brief initial blocking phase is followed by a simple exponential filling phase. The consequent kink in the packing rate would cause conventional solution methods to fail. The kink appears because high aspect ratio particles initially have a large blocking effect but fill little space. Such a kink is therefore likely to appear in other similar problems.

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[1] M. Penrose, Commun. Math. Phys. **218**, 153 (2001).
[2] J. Conway and N. Sloane, *Sphere Packings, Lattices and Groups* (Springer-Verlag, Berlin, 1993).
[3] J. Evans, Rev. Mod. Phys. **65**, 1281 (1993).
[4] J. Talbot, G. Tarjus, P. V. Tassel, and P. Viot, Colloids Surf. **165**, 287 (2000).
[5] V. Privman, Colloids Surf. **165**, 231 (2000).
[6] A. Donev, I. Cisse, D. Sachs, E. A. Variano, F. H. Stillinger, R. Connelly, S. Torquato, and P. M. Chaikin, Science **303**, 990 (2004).
[7] A. Philipse, Langmuir **12**, 1127 (1996).
[8] S. R. Williams and A. P. Philipse, Phys. Rev. E **67**, 051301 (2003).
[9] J. Parkhouse and A. Kelly, Proc. R. Soc. London, Ser. A **451**, 737 (1995).
[10] J. D. Sherwood, J. Phys. A **30**, L839 (1997).
[11] P. Viot, G. Tarjus, S. Ricci, and J. Talbot, J. Chem. Phys. **97**, 5212 (1992).
[12] Y. Isokawa, Forma **16**, 327 (2001).
[13] D. Hull, *An Introduction to Composite Materials* (Cambridge University Press, Cambridge, England, 1996).
[14] A. Renyi, Publ. Math. Inst. Hung. Acad. Sci. **3**, 109 (1958).
[15] D. J. Burridge and Y. Mao, Phys. Rev. E **69**, 037102 (2004).
[16] C. Gan and J. Wang, J. Chem. Phys. **108**, 3010 (1998).
[17] Y. Fan and J. K. Percus, Phys. Rev. Lett. **67**, 1677 (1991).
[18] R. Dickman, J. Wang, and I. Jensen, J. Chem. Phys. **94**, 8252 (1991).
[19] A. Baram and M. Fixman, J. Chem. Phys. **103**, 1929 (1995).
[20] J. S. Wang, Colloids Surf. **165**, 325 (2000).